5. V. N. Poturaev, V. I. Dyrda, and I. I. Krush, Applied Mechanics of Rubber [in Russian], Naukova Dumka, Kiev (1975).

CALCULATION OF THE COOLING OF A LIQUID MOVING IN AN UNDERGROUND CHANNEL

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An approximate solution is obtained for a problem of the cooling of a liquid moving a channel. The problem is solved on the basis of the use of the Laplace transform and the variational method.

The first solution to the problem of the cooling of a liquid in an underground passage was first given by Van-Heerden [1]. The classical method of the Laplace transform was used to obtain the solution. The final expression for the change in the temperature of the liquid in the channel was a complex relation which included Bessel functions of the first and second kind. Practical realization of this expression presented certain difficulties, even with the aid of a computer.

It should be noted that, in all cases, use of integral transforms over time leads to solutions in the form of infinite functional series or improper integrals. Here, only the main part of these expressions is used for practical calculations. Thus, if a simple method is found for directly determining a function equivalent to the main part of the exact solution, then it may be justifiably considered an approximate method suitable for practical application. Such a method, based on the joint use of the Laplace transform and the variational method, was proposed by Tsoi [2].

Let us examine the solution of the Van-Heerden problem by this approximate method and show how much simpler the final expression for the temperature of the liquid moving in the channel appears and how much more convenient it is for practical purposes.

The mathematical formulation of the problem in generalized variables has the form

$$\frac{\partial \eta}{\partial \tau'} = a\left[\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r}\right], \tag{1}$$

$$Q\frac{\partial \Theta}{\partial x} = \lambda_{gr}\frac{\partial \eta}{\partial r}\Big|_{r=R}. \tag{2}$$

The boundary conditions

$$\Theta = \eta \quad \text{at} \quad r = R, \tag{3}$$

$$\eta = 0 \quad \text{at} \quad r = \infty, \tag{4}$$

$$\Theta = 1 \quad \text{at} \quad r = 0. \tag{5}$$

The initial conditions

$$\left.\begin{matrix}\Theta = 0\\ \eta = 0\end{matrix}\right\} \quad \text{at} \quad \tau' = 0, \tag{6}$$

where

$$\Theta = \frac{t_a - t_{gr}}{t_{0v} - t_{0gr}}; \quad \eta = \frac{t_{gr} - t_{gr}}{t_{0v} - t_{0gr}}; \quad \tau' = \tau - \frac{x}{V}; \quad Q = 0.5c_v \rho_v VR.$$

We will apply the Laplace transform in the following form [3] to problem (1)-(6):

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$$\eta(r, p) = \int_0^\infty \eta(r, \tau') \exp(-p\tau')\, d\tau'.$$

Then the above equations are rewritten as follows in the image region:

$$p\bar{\eta} = a\left[\frac{d^2\bar{\eta}}{dr^2} + \frac{1}{r}\frac{d\bar{\eta}}{dr}\right], \tag{1'}$$

$$Q\frac{d\bar{\Theta}}{dx} = \lambda_{gr}\frac{d\bar{\eta}}{dr}\bigg|_{r=R}, \tag{2'}$$

$$\bar{\Theta} = \bar{\eta} \quad \text{at} \quad r = R, \tag{3'}$$

$$\bar{\eta} = 0 \quad \text{at} \quad r = \infty, \tag{4'}$$

$$\bar{\Theta} = \frac{1}{p} \quad \text{at} \quad x = 0. \tag{5'}$$

We will find the solution of Eq. (1') by the variational method [4]. For this, we convert it to the form

$$a\frac{d}{dr}\left(r\frac{d\bar{\eta}}{dr}\right) = rp\bar{\eta}. \tag{7}$$

It is known from the theory of variational calculus that Eq. (7), a so-called self-adjoint differential equation of the second order,

$$\frac{d}{dx}(py') - qy - f = 0, \tag{8}$$

is the Euler–Lagrange equation for the functional

$$J = \int_{x_0}^{x_1} [p(x)y'^2 + q(x)y^2 + 2f(x)y]\, dx. \tag{9}$$

Having determined the function y(x), realizing the minimum of the functional (9), we find the the solution of Eq. (8). For this, we use the Ritz method.

Let $y^*(x)$ be the exact solution, realizing the minimum of the integral (9), and $J(y^*) = m$ be the value of the minimum. If we succeed in constructing the function $\bar{y}(x)$, for which the value of the integral $J(\bar{y})$ is very close to m, then $\bar{y}(x)$ will be an approximation of the actual solution. If we succeed in finding a sequence of functions $\bar{y}_n$, for which $J(\bar{y}_n) \to m$, then this sequence reduces to the solution of the function.

To actually find a functional $\bar{y}(x)$ giving the value of the integral J very close to the minimum, we will examine a system of functions dependent on several parameters:

$$y = \Phi(x, a_1, a_2, \ldots, a_n), \tag{10}$$

such that the boundary conditions of the boundary-value problem are satisfied at all values of the parameters.

We will limit the class of permissible functions to the functions of system (10) and find those functions among them that give the lowest value of the integral (9). Having substituted Eq. (10) into (9) and having performed the necessary operations of differentiation and integration, we convert J into a function of n variables $a_1, a_2, a_3, \ldots, a_n$. When we obtain the minimum of this functional, the numbers a_i should satisfy the system of equations

$$\frac{\partial J}{\partial a_k} = 0 \; (k = 1, 2, 3, \ldots, n). \tag{11}$$

Having solved this system, we obtain certain values of the parameters $a_1, a_2, \ldots, a_n$ giving the functions $J(a_1, a_2, \ldots, a_n)$ an absolute minimum. Having chosen in the system (10) a

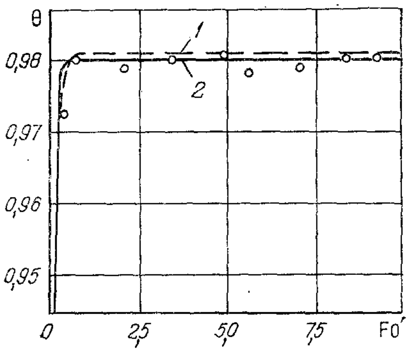

Fig. 1. Comparison of the calculated and experimental data characterizing the law of cooling of water moving along an underground channel: 1) calculation according to Van-Heerden; 2) our calculation; points denote experimental results.

function corresponding to these values of the parameters, we obtain the required approximate solution:

$$\bar{y}(x) = \Phi(x, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n). \tag{12}$$

The functional corresponding to Eq. (7) can be written thus:

$$J = \int_R^\infty \left[r \left(\frac{d\bar{\eta}}{d\,r} \right)^2 + \frac{1}{a} rp\bar{\eta}^2 \right] dr. \tag{13}$$

To find the functions η, realizing the minimum of the functional (13), we will examine a system of the form

$$\bar{\eta} = \bar{\Theta} \exp\left(1 - \frac{r}{R}\right) + \sum_{k=1}^{\infty} \bar{a}_k(p) \left\{ \exp\left[k\left(1 - \frac{r}{R}\right)\right] \left[1 - \exp\left[k\left(1 - \frac{r}{R}\right)\right]\right] \right\}, \tag{14}$$

satisfying the boundary conditions of the problem $\bar{\eta} = \bar{\Theta}$ at r = R and η = 0 at r = ∞, which is a requirement of the variational method. We will seek the distribution function of dimensionless temperature in the first approximation in the form:

$$\bar{\eta} = \bar{\Theta} \exp\left(1 - \frac{r}{R}\right) + \bar{a}_1(p) \exp\left(1 - \frac{r}{R}\right)\left[1 - \exp\left(1 - \frac{r}{R}\right)\right]. \tag{15}$$

Substituting Eq. (15) into functional (13) and differentiating and integrating with allowance for the condition $\frac{dJ}{d\bar{a}_1(p)} = 0$, we obtain an expression for the coefficient

$$\bar{a}_1(p) = -\frac{-20 + 44\frac{pR^2}{a}}{32 + 25\frac{pR^2}{a}}\bar{\Theta}. \tag{16}$$

Then the solution in images of Eq. (7) in the first approximation has the form

$$\bar{\eta} = \bar{\Theta} \exp\left(1 - \frac{r}{R}\right) - \frac{-20 + 44\frac{pR^2}{a}}{32 + 25\frac{pR^2}{a}} \bar{\Theta} \exp\left(1 - \frac{r}{R}\right) \left[1 - \exp\left(1 - \frac{r}{R}\right)\right]. \tag{17}$$

We insert the approximate solution in images for the dimensionless temperature of the ground (17) into the heat-balance equation (2') and find an expression for the dimensionless temperature of the moving liquid

$$\frac{d\overline{\Theta}}{dx}+\frac{\lambda_{\text{gr}}}{QR}\left(\frac{12+69\dfrac{pR^2}{a}}{32+25\dfrac{pR^2}{a}}\right)\overline{\Theta}=0, \tag{18}$$

the solution of which, with allowance for boundary condition (5'), will be

$$\overline{\Theta}=\frac{1}{p}\exp\left(-\frac{\xi}{\text{Bi}_Q}\,\frac{a+bp}{d+ep}\right), \tag{19}$$

where $\text{Bi}_Q=Q/\lambda_{\text{gr}}$; $\xi=x/R$; $a=12$; $b=69R^2/a$; $d=32$; $e=25R^2/a$.

The function on the right side of this expression has two features: a simple pole at the point p = 0, and an essential singular point p = −d/e. To find the original of the function (19), we will use the Laplace transform inversion formula [3] and the residue theorem [5].

To simplify subsequent calculations, we will make a variable substitution so that the essential singular point is moved to the origin of the coordinates of the complex plane. For this, we designate $q=d+ep$; $p=\dfrac{q-d}{e}$; $dp=\dfrac{1}{e}dq$ and convert Eq. (19) to the form

$$\overline{\Theta}=\frac{1}{p}\exp\left(-\frac{\xi}{\text{Bi}_Q}\,\frac{a+bp}{d+ep}\right)=\frac{F}{q-d}\exp\left(\frac{A}{q}\right), \tag{20}$$

where $A=-\dfrac{\xi}{\text{Bi}_Q}\left(a-\dfrac{d}{e}b\right)$; $B=-\dfrac{\xi}{\text{Bi}_Q}\,\dfrac{b}{e}$; $F=e\exp B$.

We will seek the original of the solution from the formula

$$f(\tau')=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\left[\frac{F}{q-d}\exp\left(\frac{A}{q}\right)\exp\left(\frac{q-d}{e}\tau'\right)\frac{1}{e}\right]dq. \tag{21}$$

In accordance with the residue theorem [5]

$$f(\tau')=\operatorname*{Res}_{q=d}\left[F(q)\exp\left(\frac{q-d}{e}\tau'\right)\frac{1}{e}\right]+\operatorname*{Res}_{q=0}\left[F(q)\exp\left(\frac{q-d}{e}\tau'\right)\frac{1}{e}\right].$$

The first residue at the point q = d is found from the expression for the residue in the case of a simple pole:

$$\operatorname*{Res}_{z=z_0}f(z)=\lim_{z\to z_0}[f(z)(z-z_0)]=\frac{F}{e}\exp\left(\frac{A}{d}\right). \tag{22}$$

The second residue at the point q = 0, an essential singular point, is found as follows. We transform the integrand in (21) and expand it into a Taylor series in the neighborhood of the point q = 0;

$$F(q)=\frac{F'}{q-d}\exp\left(\frac{A}{q}\right)\exp(q\sigma), \tag{23}$$

where $\sigma=\dfrac{\tau'}{e}$; $F'=F\dfrac{1}{e}\exp\left(-\dfrac{d}{e}\tau'\right)$. After expansion of the function into a Laurent expansion we pick out the terms containing (q^{-1}). Then the coefficient with (q^{-1}) will be the residue of the function F(q) at the point q = 0. Writing the functions comprising F(q) in the form of series and multiplying them in accordance with the Cauchy rule, after several mathematical transformations we obtain the coefficient with (q^{-1}) in the Laurent expansion of the integrand (23) at the point q = 0, which is also the residue of the integrand at this point:

$$a_{-1} = -F' \sum_{n=0}^{\infty} \left(\frac{A}{d}\right)^{n+1} \frac{e_n(\sigma d)}{(n+1)!}, \quad (24)$$

where $e_n(\sigma d)$ is a truncated exponential series,

$$e_n(\sigma d) = \sum_{k=0}^{n} \frac{(\sigma d)^k}{k!}.$$

Then the final solution for the temperature of a liquid moving in an underground channel is written in the form

$$\Theta = \exp\left(-2.76\frac{\xi}{\mathrm{Bi}_Q}\right)\left[\exp\left(2.385\frac{\xi}{\mathrm{Bi}_Q}\right) - \exp(-1.28\mathrm{Fo}') \sum_{n=0}^{\infty}\left(2.385\frac{\xi}{\mathrm{Bi}_Q}\right)^{n+1} \frac{e_n(1.28\,\mathrm{Fo}')}{(n+1)!}\right], \quad (25)$$

where $\mathrm{Fo}' = a\tau'/R^2$.

Equation (25) is an approximate solution of the problem examined here. However, it is considerably simpler than the solution obtained by Van-Heerden.

The graphical comparison of experimental data and the results of calculations with Eq. (25) and the equations of Van-Heerden [1] in Fig. 1 shows that they agree well.

NOTATION

λ_{gr}, thermal conductivity of the ground; c, specific heat; ρ, density; a, diffusivity; Q, unit quantity of heat transferred by the liquid through the channel cross section per unit of time; r, distance over radius from channel axis; $t_{gr}(r, x, \tau)$, temperature of the ground as a function of the radial and axial position and time; $t_a(x, \tau)$, temperature of the liquid; V, linear velocity of the liquid; t_0, initial temperature.

LITERATURE CITED

1. K. Van-Heerden, "Problem of transient heat flow in connection with the air cooling of coal mines," in: Problems of Heat Transfer [in Russian], Moscow—Leningrad (1959), pp. 71-74.
2. P. V. Tsoi, Methods of Calculating Individual Problems of Heat and Mass Transfer [in Russian], Energiya, Moscow (1971).
3. G. Dech, Guide to Practical Application of the Laplace Transform and Z-Transform [in Russian], Nauka, Moscow (1971).
4. R. Schechter, Variational Method in Engineering Calculations [Russian translation], Mir, Moscow (1971).
5. V. A. Ditkin and A. P. Prudnikov, Operational Calculus [in Russian], Vysshaya Shkola, Moscow (1966).

DISTRIBUTION OF THE DISPERSE FRACTION OF AN INJECTED POLYDISPERSE JET IN A GAS FLOW

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The distribution of a polydisperse droplet jet over a gas flow is theoretically investigated. Results are given for specific nozzles.

Questions of the distribution of a disperse condensed phase over a gas flow are of importance in a whole series of processes of chemical engineering, the cooling of hot gases, combustion, etc. As a rule, this phase is introduced in individual regions using a dispers-

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